United Kingdom
Mathematics Trust

Intermediate Mathematical Olympiad

Cayley paper

supported by

Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

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1. In the four-digit number 4753, three two-digit numbers are formed by successive pairs of digits $(47, 75, 53)$. Exactly two of these two-digit numbers are prime. Find all four-digit numbers in which all four digits are prime, and **all three** two-digit numbers formed by successive digits are prime.

SOLUTION

Let P represent such a four-digit number.

As the digits of P must each be prime, the only choices of digits we can make are 2, 3, 5 or 7.

It will be helpful to list the two digit prime numbers which use only these digits: 23, 37, 53, and 73.

The first two digits of P must form a prime number when taken as a two-digit number (from left to right), so we should consider each of the four options in turn.

If the first two digits are 2 and 3 respectively, this means that the third digit must be 7 as the second and third digits must form a two-digit prime number and our only choice is 37. The final digit must be 3, as the third and fourth digits must also form a two-digit prime number and our only choice is 73. This gives 2373 as a possible value for P.

If we consider the other three options for the first two digits in turn, similar reasoning yields 3737, 5373 and 7373 as possible values for P.

In conclusion, we have four possible values for P which are: 2373, 3737, 5373, and 7373.

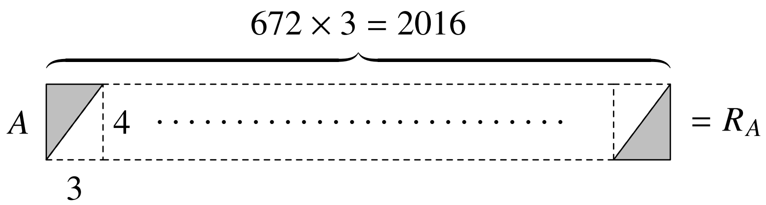
> **2.** Jack has a large number of tiles, each of which is in the shape of a right-angled triangle with side lengths 3 cm, 4 cm and 5 cm. Is it possible for Jack to combine a number of these tiles, without gaps or overlap, to form a rectangle of size 2016 cm by 2021 cm?

SOLUTION

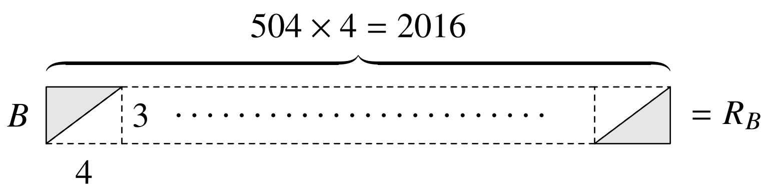
Throughout this solution whenever a rectangle is described as having size x cm by y cm, the first and second lengths refer to the top and bottom edges and the left and right edges respectively.

Two of the right-angled triangular tiles can be joined together along their longest sides to form a rectangle. This rectangle can have size 3 cm by 4 cm or 4 cm by 3 cm, which we will refer to as A and B respectively. We will consider A and B to be new tiles that we can use.

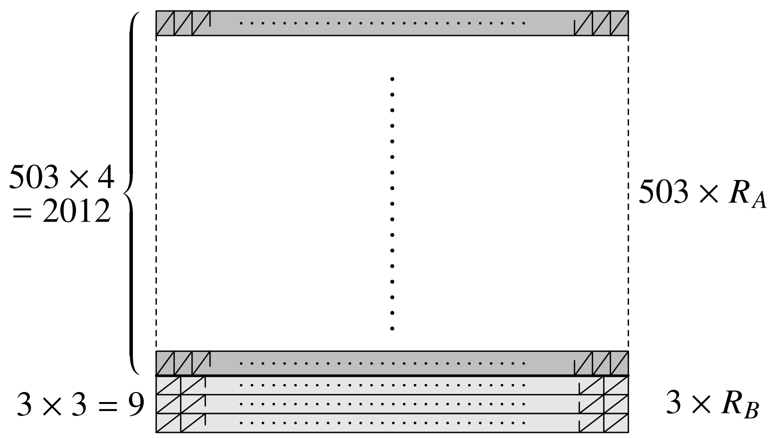
We can place 672 A tiles side by side to create a rectangle of size 2016 cm by 4 cm. We will refer to such a rectangle as R_A and will consider it a new tile that we can use.



We can place 504 B tiles side by side to create a rectangle of size 2016 cm by 3 cm. We will refer to such a rectangle as R_B and will consider it a new tile that we can use.



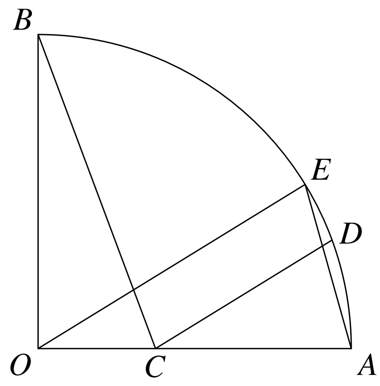
If we stack 503 R_A tiles and 3 R_B tiles on top of each other we will create a 2016 cm by 2021 cm rectangle.



This means that Jack's task is possible.

3. In the diagram, OAB is a quarter circle, OE and CD are parallel, BC and EA are parallel, and $\angle BCD = 4 \times \angle OBC$.

What is the size of $\angle OBC$?

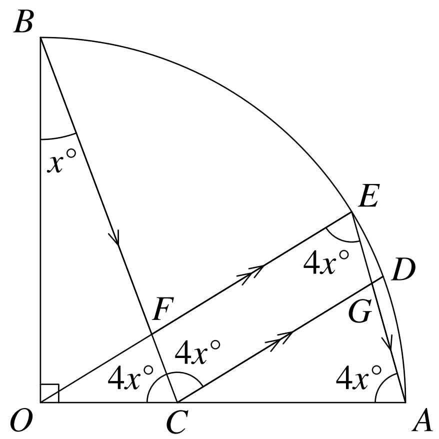


SOLUTION

We begin by defining F as the intersection of OE and BC and G as the intersection of CD and EA.

Let $\angle OBC = x°$.

Quadrilateral $FCGE$ must be a parallelogram, as OE is parallel to CD and BC is parallel to EA (given in the question).



We have that $\angle BCD = \angle AEO = 4x°$, as opposite angles in a parallelogram are equal and we are told in the question that $\angle BCD = 4 \times \angle OBC$.

Triangle OAE is isosceles as $OA = OE$ because both are radii of the quarter circle OAB, thus $\angle OAE = \angle AEO = 4x°$.

Also $\angle OCB = \angle OAE$, as BC is parallel to EA and hence they are corresponding angles.

As OAB is a quarter circle then $\angle BOA = 90°$.

Considering the sum of the interior angles of the right-angled triangle OCB,

$$180° = 90° + 4x° + x°$$

Finally, the previous equation simplifies to $90° = 5x°$ and so $\angle OBC = x° = 18°$.

4. Let $S(n)$ denote the sum of the first n terms of the series

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots \quad .$$

For example, $S(5) = 1 - 2 + 3 - 4 + 5 = 3$.

For what values of a and b does $S(a) + S(b) + S(a+b) = 1$?

SOLUTION

It is helpful to consider the terms of $S(n)$ in pairs from left to right, where we pair up the first and second terms, the third and fourth terms, etc.

Each pair is of the form $(i) - (i+1) = -1$ for some positive whole number i.

This observation allows us to find closed form expressions for $S(n)$.

If n is even, every term will be part of a pair and $S(n) = (-1) \times (\frac{n}{2}) = \frac{-n}{2}$.

If n is odd, every term but the final term will be part of a pair and $S(n) = (-1) \times (\frac{n-1}{2}) + n = \frac{n+1}{2}$.

To find the solutions to the equation $S(a) + S(b) + S(a+b) = 1$, we will consider three separate cases:

(i) a and b both even.

If a and b are both even then so is $a + b$. This means that $S(a)$, $S(b)$ and $S(a+b)$ are each negative and so cannot sum to one.

(ii) a and b both odd.

If a and b are both odd then $a + b$ is even. This means that,

$$S(a) + S(b) + S(a+b) = \frac{a+1}{2} + \frac{b+1}{2} + \frac{-(a+b)}{2}.$$

The right-hand side simplifies to 1 for all values of a and b.

(iii) a and b have opposite parity.

If a and b have opposite parity then $a + b$ is odd. Without loss of generality we may assume that a is odd and b is even, which means that,

$$S(a) + S(b) + S(a+b) = \frac{a+1}{2} + \frac{-b}{2} + \frac{a+b+1}{2}.$$

The right-hand side simplifies to $a + 1$ which can only be equal to one if $a = 0$, but we know that a is odd (and $a > 0$).

The equation $S(a) + S(b) + S(a+b) = 1$ is satisfied precisely when a and b are both odd numbers.

5. Real numbers p, q, r, x, y, z satisfy the equations

$$\tfrac{x}{p} + \tfrac{q}{y} = 1,$$

$$\tfrac{y}{q} + \tfrac{r}{z} = 1.$$

Prove that $pqr + xyz = 0$.

SOLUTION

We start by noting that none of p, q, y or z are zero, as they each appear as a denominator in the given system of equations. This means that we can multiply and divide by these variables without introducing inconsistencies or extra solutions to the given system of equations.

We shall label the given equations as:

$$\frac{x}{p} + \frac{q}{y} = 1; \tag{1}$$

$$\frac{y}{q} + \frac{r}{z} = 1. \tag{2}$$

Multiplying the first equation by py and the second equation by pqz yields:

$$xy + pq = py; \tag{3}$$

$$pyz + pqr = pqz. \tag{4}$$

Using equation (3) we can replace the pyz term in equation (4) with $(xy + pq)z$ to obtain $(xy + pq)z + pqr = pqz$.

After expanding the bracket we have $xyz + pqz + pqr = pqz$, and then subtracting pqz from both sides yields $xyz + pqr = 0$.

Interchanging the terms on the left-hand side yields $pqr + xyz = 0$, which is the result we were asked to prove.

6. During an early morning drive to work, Simon encountered n sets of traffic lights, each set being red, amber or green as he approached it. He noticed that consecutive lights were never the same colour.

Given that he saw *at least* two red lights, find a simplified expression, in terms of n, for the number of possible sequences of colours Simon could have seen.

SOLUTION

We will find three separate expressions, each relating to sequences which satisfy the condition that consecutive lights were never the same colour:

(i) T_n= Total number of possible sequences of length n.

Consider the first colour encountered, it can be red, amber or green. This gives three choices for the first colour.

Next consider the second colour encountered, this must not be the same as the first colour but could be either of the other two colours. This gives two choices for the second colour.

Considering subsequent colours is much the same as considering the second, the next colour in the sequence cannot be the same as the previous colour but can be one of the other two.

This means that for the first colour there are three choices and for each colour after the first there are two choices and so $T_n = 3 \times 2^{n-1}$.

(ii) R_{0_n}= Number of possible sequences, of length n, which contain zero red lights.

Consider the first colour encountered, it cannot be red so must be amber or green. This gives two choices for the first colour.

If the first colour is amber, the second must be green, the third amber and so on. If the first colour is green, the second must be amber, the third green and so on.

This means that $R_{0_n} = 2$, which is independent of n and thus we will refer to it as R_0.

(iii) R_{1_n}= Number of possible sequences, of length n, which contain exactly one red light.

If red appears as the first or last colour encountered then we must have a sub-sequence of length $(n - 1)$ which contains no red lights. There are $R_0 = 2$ of these in each case, so $2 \times R_0 = 4$ possible sequences.

If red appears elsewhere, then either side of it we must have sub-sequences which contain no red lights. This means for each possible position of the colour red we have R_0 choices for each of the two sub-sequences, and so $R_0 \times R_0 = 4$ possible sequences of length n. As there are $n - 2$ positions (which are not first or last) this gives $4(n - 2)$ possible sequences.

This means that $R_{1_n} = 4 + 4(n - 2) = 4n - 4$.

Finally, we have that the number of possible sequences containing at least two red lights is $T_n - R_0 - R_{1_n} = 3 \times 2^{n-1} - 2 - (4n - 4)$ which simplifies to $3 \times 2^{n-1} - 4n + 2$.